Boson Star Rotation: A Newtonian Approximation

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Abstract

Using the Newtonian approximation, we study rotating compact bosonic objects. The equations which describe stationary states with non-zero angular momentum are constructed and some numerical results are presented as examples. Limits on the applicability of the Newtonian approximation are discussed.

1 Introduction

Boson stars are well known gravitionally bound states of complex scalar fields. These objects were first studied by Ruffini and Bonazolla [1], and, since then, a large number of papers on this subject have been published [2], including three recent reviews [3]. As pointed out by Ferrell and Gleiser [4], we have two distinct motivations to study boson stars. On one hand, they make a good laboratory to study compact self gravitating objects and to explore the differences and similarities between then and the usual stars. On the other hand, considering that scalar fields play an important role in theories of fundamental forces, it is reasonable to address the question of formation and stability of bosonic compact objects, including also the study of mixed boson-fermion objects [5], which may be even more probable to find in nature.

Concerning the analysis of stable solutions of boson stars, most of the work done is restricted to spherically symmetric configurations, including the ground state and excitations which are also spherically symmetric. Among the exceptions, there is the work of Ferrell and Gleiser [4], which analyses the emission of gravitational radiation by boson stars in the Newtonian approximation. The compact object is supposed to decay from an excited state with non zero angular momentum to the ground state with the emission of gravitational waves. The excited state is treated as a pertubation in which the excited bosonic particles, carrying the angular momentum, move in the background potential of the spherically symmetric state.

Recently, Kobayashi, Kasai and Futamase [6] consider the slow rotation of a relativistic bosonic star. They look for slowly rotating solutions which are similiar to the rotation of conventional objects, following the approach of

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Hartle [7]. They conclude with a negative result, namely that the relativistic boson star has no stationary solutions with slow rotation. However, the possibility of rapid rotations, which can not be treated pertubatively, is not excluded.

The failure to describe rotational states pertubatively is itself instructive. First order pertubation theory is well suited to describe pertubative states which can be obtained by continuos deformations of the ground state and, to keep it first order, we must also require that the pertubation parameter is "small" in some sense. If the excited rotational states we are searching for form a discrete set, they can not be obtained by pertubation theory, even if they do not rotate at relativistic speeds. And up to now, we do not known whether this compact objects always rotate at relativistic speeds or not.

In this paper, we return to the problem of rotating boson stars and, in an attempt to avoid the pertubative approach, we consider the Newtonian approximation, which is known to be valid for boson stars provided that the central density or the total mass of the star is not higher than a certain critical limit [4]. We assume the hypothesis that, at least for the first excited states, the rotational effect may be well described by the non-relativistic theory. This is analogous to the excited states of atoms which are not obtained as pertubations of the ground state but, even so, are well described by the non-relativistic theory. Under this assumption, the boson star is analysed, and we search for stationary solutions with non-zero angular momentum, following the approach of [4], with one major difference: the excited sates of the scalar field and their gravitational potential are computed simultaneously in a coupled system of equations, without any reference to a fixed spherically symmetric background. The resulting solutions describe stationary rotational states which are not pertubative deformations of the ground state. At the

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end, we check the validity of the Newtonian approximation and we identify the limits which justify the use of the non-relativistic approximation.

2 Newtonian Approximation for Boson Stars

We consider complex scalar fields which are coupled to gravity only. The action is given by:

$$S = \int d^4x \ (\frac{R}{16\pi G} + g^{\mu\nu}\partial_\mu\Phi^*\partial_\nu\Phi - M^2\Phi^*\Phi) \tag{1}$$

Since we will consider only the weak gravity limit of general relativity, the metric is expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu} = diag(1, -1, -r^2, -r^2\sin^2(\theta))$. Using the weak field approximation of General Relativity [8], we get

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{2}$$

where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$ and

$$T_{\mu\nu} = \partial_\mu\Phi^*\partial_\nu\Phi + \partial_\nu\Phi^*\partial_\mu\Phi - \eta_{\mu\nu}(\eta^{\alpha\beta}\partial_\alpha\Phi^*\partial_\beta\Phi - M^2\Phi^*\Phi) \tag{3}$$

The equation for Φ, derived from (1), is:

$$\Box\Phi + M^2\Phi = 0 \tag{4}$$

We will search for solutions with stationary rotation so Φ will be allowed to depend on t and φ only through a phase, $\Phi(\vec{r}, t) = \phi(r, \theta)e^{iwt}e^{im\varphi}$. $T_{\mu\nu}$, given by (3), will be independent on both t and φ, and so will be the metric. To take into account deformations produced by the rotation, we allow both ϕ and $h_{\mu\nu}$ to depend on r and θ.

Finally, we will restrict ourselves to cases in which special relativistic effects are not important. The constraint imposed by this restriction will be checked to be consistent later on this paper. In this approximation, the only relevant component of $h_{\mu\nu}$ is $h_{00} = 2V(r,\theta)$, where $V(r,\theta)$ is the Newtonian potential. Equations (2) and (4) become :

$$\vec{\nabla}^2 V = 8\pi G M^2 \phi^2 \tag{5}$$

$$- D^2\phi - (w^2 - M^2)\phi + 2w^2 V\phi = 0 \tag{6}$$

where $D^2\phi = (\vec{\nabla}^2 - \frac{m^2}{r^2 \sin^2\theta})\phi$, $V = V(r,\theta)$ and $\phi = \phi(r,\theta)$. In the non-relativistic limit, the gravitational binding energy E per particle must be much smaller than M, and the scalar field frequency may be written as $w = E + M$ with $|E| \ll M$. The scalar field equation reduces to a Schrodinger equation:

$$- \frac{1}{2M}\vec{\nabla}^2\phi + MV\phi = E\phi \tag{7}$$

Equations (5) and (7) must be solved subjected to the charge conservation constraint. The gauge invariance of the complex scalar field implies the conservation of $j^\mu = i(\partial^\mu \phi \phi^* - \phi \partial^\mu \phi^*)$ with conserved particle number:

$$N = 2M \int \phi^2 r^2 \, dr \, \sin(\theta) \, d\theta \, d\varphi \tag{8}$$

3   Stationary Solutions

We now look for solutions of (5), (7) with non-zero angular momentum described by $\Phi = e^{iwt}e^{im\varphi}\phi(r,\theta)$ and $V = V(r,\theta)$. We expand $\phi(r,\theta)$ in associated Legendre functions:

$$\phi(r,\theta) = \frac{1}{\sqrt{4\pi}} \sum_{l=m}^{\infty} R_l(r) P_l^m(\theta) \tag{9}$$

and, since V, differently from Φ, has no φ dependence, we consider:

$$V(r,\theta) = \sum_{l=0}^{\infty} V_l(r) P_l(\theta) \tag{10}$$

With (9),(10) and the orthogonality relations of P_l^m, we rewrite (5),(7) as a larger system of equations, which contains one equation for each value of l:

$$V_{l_0}'' + \frac{2}{r} V_{l_0}' - \frac{l_0(l_0+1)}{r^2} V_{l_0} = GM^2(2l_0+1) \sum_{l,l'} A_{l\, l'\, l_0} R_l R_{l'} \tag{11}$$

$$\frac{1}{2M} \left[R_{l_0}'' + \frac{2}{r} R_{l_0}' - \frac{l_0(l_0+1)}{r^2} R_{l_0} \right] + E R_{l_0} =$$

$$M \frac{(2l_0+1)}{2} \frac{(l_0-m)!}{(l_0+m)!} \sum_{l=m}^{\infty} \sum_{l'=m}^{\infty} A_{l\,l_0\,l'} R_l V_{l'} \tag{12}$$

with $\quad' = \partial_r \quad$ and

$$A_{l\,l'\,l_0} = \int_{-1}^{1} dx\, P_l^m(x)\, P_{l'}^m(x)\, P_{l_0}(x) \tag{13}$$

This system of equations may be rescaled by introducing the new variables [4]:

$$\hat{r} = \hat{N} M r,$$

$$V(r,\theta) = \hat{N}^2 \hat{V}(\hat{r},\theta),$$

$$R(r) = \hat{N}^2 (2G)^{-1/2} \hat{R}(\hat{r}),$$

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$$E = M\hat{N}^2\hat{E}$$

and

$$\hat{N} = GM^2N\frac{(2l+1)(l-m)!}{(l+m)!} \tag{14}$$

With this new variables, our system can be written as:

$$\hat{V}_{l_0}'' + \frac{2}{\hat{r}}\hat{V}_{l_0}' - \frac{l_0(l_0+1)}{\hat{r}^2}\hat{V}_{l_0} = \frac{(2l_0+1)}{2}\sum_{l,l'}A_{l\,l'\,l_0}\hat{R}_l\hat{R}_{l'} \tag{15}$$

$$\frac{1}{2}\left[\hat{R}_{l_0}'' + \frac{2}{\hat{r}}\hat{R}_{l_0}' - \frac{l_0(l_0+1)}{\hat{r}^2}\hat{R}_{l_0}\right] + \hat{E}\hat{R}_{l_0} =$$

$$\frac{(2l_0+1)}{2}\frac{(l_0-m)!}{(l_0+m)!}\sum_{l=m}^{\infty}\sum_{l'=m}^{\infty}A_{l\,l_0\,l'}\hat{R}_l\hat{V}_{l'} \tag{16}$$

where $'$ now stands for $\frac{\partial}{\partial\hat{r}}$ and we must add the normalization condition derived from (8):

$$\int\hat{R}_{l_0}^2(\hat{r})\,\hat{r}^2\,d\hat{r} = 1 \tag{17}$$

We may now systematically search for solutions of (15), (16) for different values of l and m. The basic idea is to consider solutions with $\hat{R}_L \neq 0$ for one particular value of l, L, with $\hat{R}_l$ for $l \neq L$. The ground state equations correspond to the choice $m = 0$ with $\hat{R}_l = 0$ for $l \neq 0$, in which case the equations simplify to:

$$\hat{V}_l'' + \frac{2}{\hat{r}}\hat{V}_l' = \frac{1}{2}A_{0\,0\,l}(\hat{R}_0)^2$$

$$\frac{1}{2}\left[\hat{R}_0'' + \frac{2}{\hat{r}}\hat{R}_0'\right] + \hat{E}\hat{R}_0 = \frac{1}{2}\sum_{l'=0}^{\infty}A_{0\,0\,l'}\hat{R}_0\hat{V}_{l'} \tag{18}$$

and $A_{00l} = \int_{-1}^{1} dx\, P_l(x) = 2\delta_{l0}$. For $l \neq 0$, the equation for $\hat{V}_l$ is homogeneous and the solution is the trivial $\hat{V}_l(\hat{r}) = 0$. In the expansion (10), the only non-zero component is $\hat{V}_0$ and we are left with the simple system:

$$\hat{V}_0'' + \frac{2}{\hat{r}}\hat{V}_0' = (\hat{R}_0)^2$$

$$\frac{1}{2}\left[\hat{R}_0'' + \frac{2}{\hat{r}}\hat{R}_0'\right] + \hat{E}\hat{R}_0 = \hat{R}_0\hat{V}_0 \tag{19}$$

The excited states will correspond to other choices of $\hat{R}_L \neq 0$. As an example, we may focus on the $l = 2$, $m = 0$ state described by the system of equations (15), (16) with the appropriate values of l and m. We set $\hat{R}_l = 0$ for $l \neq 2$. The r.h.s of equations (15) are given by $\frac{2l_0+1}{2}A_{22l_0}(\hat{R}_2(\hat{r}))^2$ where $A_{22l_0} = \int_{-1}^{1} dx\,(P_2(x))^2\,P_{l_0}(x)$, with the following numerical values: $A_{220} = 2/5$, $A_{222} = 4/35$, $A_{224} = 4/35$, $A_{22l} = 0$ for $l > 4$ and for odd values of l. So, for $l = 2$ and $m = 0$, the Newtonian potential will be given by:

$$V(r,\theta) = V_0(r) + V_2(r)\,P_2(\theta) + V_4(r)\,P_4(\theta) \tag{20}$$

For $l > 4$, V_l, as solution of an homogeneuos system, may be set equal to zero. V_0, V_2, V_4 and R_2 will be given as solutions of:

$$\hat{V}_0'' + \frac{2}{\hat{r}}\hat{V}_0' = \frac{1}{5}(\hat{R}_2)^2 \tag{21}$$

$$\hat{V}_2'' + \frac{2}{\hat{r}}\hat{V}_2' - \frac{6}{\hat{r}^2}\hat{V}_2 = \frac{2}{7}(\hat{R}_2)^2 \tag{22}$$

$$\hat{V}_4'' + \frac{2}{\hat{r}}\hat{V}_4' - \frac{20}{\hat{r}^2}\hat{V}_4 = \frac{18}{35}(\hat{R}_2)^2 \tag{23}$$

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$$\frac{1}{2}\left[\hat{R}_2'' + \frac{2}{\hat{r}}\hat{R}_2' - \frac{6}{\hat{r}^2}\hat{R}_2\right] + \hat{E}\hat{R}_2 = \hat{R}_2\left[\hat{V}_0 + \frac{2}{7}\hat{V}_2 + \frac{2}{7}\hat{V}_4\right] \qquad (24)$$

with the appropriate boundary conditions. For V_0, V_2 and V_4, we may incorporate these boundary conditions by formally solving (21),(22),(23) with the help of Green's functions. As usual, we consider the solution of:

$$G_l''(r,r') + \frac{2}{r}G_l'(r,r') - \frac{l(l+1)}{r^2}G_l(r,r') = \frac{1}{r^2}\delta(r-r') \qquad (25)$$

given by :

$$G_l(r,r') = -\frac{1}{(2l+1)}\frac{r_<^l}{r_>^{l+1}} \qquad (26)$$

which is regular at $r \to 0$ and goes to zero for $r \to \infty$.

In the particular case we are considering, namely, $l = 2$ and $m = 0$, we end up with:

$$\hat{V}_0(r) = -\frac{1}{5}\left[\frac{1}{\hat{r}}\int_0^{\hat{r}} dr'\,(r')^2\,(\hat{R}_2(r'))^2 + \int_{\hat{r}}^\infty dr'\,r'(\hat{R}_2(r'))^2\right] \qquad (27)$$

$$\hat{V}_2(r) = -\frac{2}{35}\left[\frac{1}{\hat{r}^3}\int_0^{\hat{r}} dr'\,(r')^4\,(\hat{R}_2(r'))^2 + \hat{r}^2\int_{\hat{r}}^\infty dr'\,\frac{1}{r'}\,(\hat{R}_2(r'))^2\right] \qquad (28)$$

$$\hat{V}_4(r) = -\frac{2}{35}\left[\frac{1}{\hat{r}^5}\int_0^{\hat{r}} dr'\,(r')^6\,(\hat{R}_2(r'))^2 + \hat{r}^4\int_{\hat{r}}^\infty dr'\,\frac{1}{(r')^3}\,(\hat{R}_2(r'))^2\right] \qquad (29)$$

and we are left with just one (integral-differential) eigenvalue equation for $\hat{R}_2(\hat{r})$:

$$\frac{1}{2}\left[\hat{R}_2'' + \frac{2}{\hat{r}}\hat{R}_2' - \frac{6}{\hat{r}^2}\hat{R}_2\right] - \hat{R}_2 W(\hat{r}) = -\hat{E}_2\hat{R}_2 \qquad (30)$$

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with

$$W(\hat{r}) = \hat{V}_0(\hat{r}) + \frac{2}{7}\hat{V}_2(\hat{r}) + \frac{2}{7}\hat{V}_4(\hat{r}) \,, \tag{31}$$

$\int_0^\infty d\hat{r}\, \hat{r}^2\, (\hat{R}_2(\hat{r}))^2 = 1$, and $\hat{R}_2(0) = 0$.

The procedure described in the $l = 2$, $m = 0$ example may be applied to all excited states. We should note here that the infinite sums in the r.h.s. of (15), (16) will always reduce to finite sums because $\left[P_{l_0}^m(x)\right]^2$ is a polynomial of order l_0^2, and may be expanded as a sum of Legendre polynomials with $l \leq 2\,l_0$. So, using the orthogonality of the Legendre polynomials, for any fixed value of l_0, $A_{l_0\, l_0\, l} = 0$ for $l > 2\,l_0$.

Without any intention to make a complete analysis of these excited states, we present the numerical results for $l = 0$, $m = 0$, the groundstate, and for the excited states $l = 1$, $m = 0$; $l = 1$, $m = 1$; $l = 2$, $m = 0$. Basically, we start with a trial function, $R^{i=0}(\hat{r})$, satisfying the appropriate boundary conditions. The index i gives the iteration order. We construct $W(\hat{r})$ using (31), or the equivalent expression for other states, and we use it to obtain a new $R^{i=1}(\hat{r})$, with the corresponding eigenvalue $E^{i=1}$. The process is then repeated with $R^{i=1}$, and so on, until convergence is achieved in the solutions and, up to a given precision, no effect is introduced by new iterations. Our results are plotted in the figures 1, 2, 3. Note that the excited states are qualitatively different from the ground state and this is why they can not be obtained as simple pertubations of the ground state.

4 Conclusion

Following the prescription outlined here, in principle, we may construct all the excited levels of the rotating boson star, and based on these states, pre-

dictions can be made about the spectrum produced by boson stars decaying from excited states to states with lower energy. However, we still must check the consistence of the non-relativistic approximation made at the beginning.

Basically, we must require that the gravitational field is weak and that no relativistic speeds are present. The weakness of the field is verified by requiring that the mass of the star in the Newtonian approximation is only slightly affected by rotation. The total mass of each state, M_T, is given by [4]:

$$M_T = M\,N + N\,E = \hat{N}(\frac{M_{Planck}^2}{M})(1 + \hat{E}\hat{N}^2) \qquad (32)$$

The rotation of the star changes its mass by the relative amount:

$$\Delta = \frac{M_T^{excited} - M_T^0}{M_T^0} = \frac{(\hat{E}_{excited}\hat{N}^2 - \hat{E}_0\hat{N}_0^2)}{1 + \hat{E}_0\hat{N}_0^2} \qquad (33)$$

where we are comparing states with the same total number N of particles, and N is related to $\hat{N}_0$ and $\hat{N}$ by (14). To be consistent with the Newtonian approximation, we should only apply the above results for boson stars with $\Delta \ll 1$. Since $|\hat{E}_0| \sim 10^{-1}$ and $|\hat{E}_{excited}| < |\hat{E}_0|$, $\Delta \ll 1$ is always satisfied by $\hat{N}_0^2 \ll 1$. For $\hat{N}_0^2 \sim 10^{-2}$, the boson star has a mass of order $10^{10}\,kg$, too small to resemble a conventional stellar object. However, this number is not much different from the maximum mass of boson stars of free scalars in the ground state, which is roughly $10^{11}\,kg$.

The restriction coming from the small velocities limit is more severe. To check for relativistic speeds, we assume that a global effective angular velocity Ω^{eff} may be associated to the stationary excited state, with:

$$L_z^{eff} = I\,\Omega^{eff} \quad and \quad I = \int \rho\,z^2\,dV \qquad (34)$$

and ρ is the energy density given by T_{00}. Taking into account that $I > I_0$ where I_0 is the moment of inertia of the ground state configuration, we have:

$$L_z^{eff} > I_0 \ \Omega^{eff} = \frac{2}{3} < r_0^2 > M_0^T \ \Omega^{eff} \tag{35}$$

with $< r_0^2 > = \frac{\int \rho_0 r^2 dV}{\int \rho_0 dV} = \frac{1}{M_0^T} \int \rho_0 r^2 dV$.

This effective angular momentum is then identified to the angular momentum of the configuration obtained by direct integration of the angular momentum tensor:

$$L_z = \int (x T_{0y} - y T_{ox}) = mw \frac{N}{M} \tag{36}$$

Taking $m = l$ and comparing (35) and (36), we have:

$$\Omega^{eff} \ < \ \frac{3l}{2M < r_0^2 >} \frac{w}{w_0} \tag{37}$$

and $v^{max} \sim 2 < r > \Omega^{eff}$ satisfies :

$$v^{max} < 3 \ \frac{< \hat{r} >}{< \hat{r}_0^2 >} \frac{w}{w_0} \ \hat{N}_0 \ \frac{l(l+m)!}{(2l+1)(l-m)!} \tag{38}$$

A sufficient condition to guarantee the use of non-relativistic approximation is:

$$3 \ \frac{< \hat{r} >}{< \hat{r}_0^2 >} \frac{w}{w_0} \ \hat{N}_0 \ \frac{l(l+m)!}{(2l+1)(l-m)!} \ll 1. \tag{39}$$

Now, $w \sim w_0$, but $< \hat{r} >$ grows fast with the excitation levels and even faster grows the factor $\frac{l(l+m)!}{(2l+1)(l-m)!}$.

Our conclusion is that the non-relativistic rotation is possible only for very small bosonic objects (small values for $\hat{N}_0$) and, even starting with small $\hat{N}_0$, only the first excited states will be well describe by the Newtonian

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theory. For higher excited states and objects with $\hat{N}_0 \sim 1$, relativistic effects will be important and we can rely only on a fully relativistic calculation. These restrictive limits tell us that the numerical results of our calculation can not be applied to large, astrophysically important stellar objects. Even so, we showed that small compact bosonic objects rotate producing a discrete set of excited states and we set limits on the applicability of the Newtonian approximation. Since the introduction of self-coupling is known to increase the maximum mass, it would be interesting to study how the self-coupling changes the rotational states, a problem we hope to address in the future.

5   Acknowledgments

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Figure Captions

Fig. 1a. Ground state ($l = 0$, $m = 0$) function, $\hat{R}(\hat{r})$, normalized to $\int_0^\infty d\hat{r}\, \hat{r}^2\, (\hat{R}(\hat{r}))^2 = 1$, for the last twelve iterations. $\hat{E}_{l=0,m=0} = -0.16$.

Fig. 1b. Newtonian potential for each iteration in the same case ($l = 0$, $m = 0$).

Fig. 2. Excited states also normalized to $\int_0^\infty d\hat{r}\, \hat{r}^2\, (\hat{R}(\hat{r}))^2 = 1$. Curve 1: $l = 1$, $m = 1$, $\hat{E}_{1,1} = -0.025$; Curve 2: $l = 1$, $m = 0$, $\hat{E}_{1,0} = -0.0071$; Curve 3: $l = 2$, $m = 0$, $\hat{E}_{2,0} = -0.0013$.

Fig. 3. Newtonian potential for $l = 2$, $m = 0$, showing contributions from $\hat{V}_0(\hat{r})$, $\hat{V}_2(\hat{r})$ and $\hat{V}_4(\hat{r})$. The potential well, with constant, non-zero value for $r \to 0$ is $\hat{V}_0(\hat{r})$. The other two represent $\hat{V}_2(\hat{r})$ and $\hat{V}_4(\hat{r})$.